Elementary calculus of modern financial mathematics [4]

Addenda: errata, extra exercises, additional material

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Contents

Chapter 1

Financial indices appear to be stochastic processes

Exercises

1.1. Get your friends and family to play this more interactive game that illustrates key aspects of stochastic mathematics in application to finance.

1. Mark out on a big sheet of paper, a sequence of 30 squares, and label them consecutively 0 (bankrupt), 1, 1, 2, 2, 3, 3, 4, . . . , 14, 14, 15 (millionairedom). Each of these squares represents one state in a 30 state Markov chain. Imagine each state represents the value of some asset such as the value of a small business that each player is managing.
2. Give each player a token and a six sided die.
3. At the start place each token on the second "2", the fifth state. Imagine this corresponds to the small business having an initial value of $200,000.

4. Each turn in the game corresponds to say one year in time. In each year business may be poor or may grow depending upon how other businesses operate. Thus, in each turn,
   (a) on the count of three, each player extends a hand showing either one finger or no fingers;
   (b) the players who are in the minority then move up some states (that is, if the number of players who show one finger is more than those showing none, then the players showing none move up some states, and vice versa);
   (c) the players who are in the majority roll their die and move down some states if they roll $1, \ldots, 4$, and stay in the same state upon rolling 5 or 6.

   But the number of states (squares) they move is given by the number written in each square. Thus in the first move, because the fifth square/state is has a "2": a player moving up moves from the fifth square to the seventh square; and a player moving down moves to the third square.

   That the number written in each square is (roughly) proportional to the position of the square in the sequence corresponds to the financial reality that small businesses usually grow/shrink by small amounts, whereas big companies grow/shrink by big amounts. Investors expect returns in proportion to their investment.
5. Each player continues to role their die and move until they reach 0 or 15. That is, imagine they continue to operate their business until they either go bankrupt or reach millionairedom.

Questions:

1. Why might you *expect* each business to grow? that is,

why might you *expect* each player to reach the 'millionairedom' state?

2. When you play it, roughly what proprotion of players do reach 'millionairedom'? and what proportion go 'bankrupt'?
3. How do you explain the actual results?

Chapter 2

Ito's stochastic calculus introduced

Example 2.1 (Interpret SDEs carefully) What if one interprets the SDE $dX = \beta X\, dW$ as the difference equation $\Delta X_n = \beta X_{n+1} \Delta W_n$? We proceed similarly by rearranging for X_{n+1}/X_n:

$$\begin{aligned} X_{n+1} - X_n &= \beta X_{n+1} \Delta W_n \\ \Rightarrow \frac{X_{n+1}}{X_n} - 1 &= \frac{X_{n+1}}{X_n} \beta \Delta W_n \\ \Rightarrow \frac{X_{n+1}}{X_n}(1 - \beta \Delta W_n) &= 1 \\ \Rightarrow \frac{X_{n+1}}{X_n} &= \frac{1}{1 - \beta \Delta W_n} . \end{aligned}$$

Now recall the example solution for a multiplicative noise relies on the observation

$$\Delta \log X_n = \log X_{n+1} - \log X_n = \log \frac{X_{n+1}}{X_n} .$$

Here, then

$$\begin{aligned}\Delta \log X_n &= \log \frac{1}{1-\beta\Delta W_n} \\ &= -\log(1-\beta\Delta W_n) \\ &= \beta\Delta W_n + \tfrac{1}{2}\beta^2\Delta W_n^2 + \cdots\end{aligned}$$

Recalling that after summing many small increments, on relatively large scales $\Delta W^2 = \Delta t$ so summing both sides of the above gives

$$\log X_n - \log X_0 = \beta(W_n - W_0) + \tfrac{1}{2}\beta^2(t_n - t_0).$$

Using $t_0 = W_0 = 0$ and $t_n = t$ then

$$X(t) = X_0 \exp\left[\tfrac{1}{2}\beta^2 t + \beta W(t)\right].$$

In contrast to the standard definition, this predicts deterministic growth! *One must be careful to use the standard 'Ito interpretation' of stochastic differential equations.* ∎

Exercises

2.1. What if one interprets the SDE $dX = \beta X\,dW$ as the difference equation $\Delta X_n = \beta\frac{1}{2}(X_{n+1} + X_n)\Delta W_n$? Deduce that the solution of the SDE is the so-called 'Stratonovich interpretation' $X(t) = X_0 \exp\left[\beta W(t)\right]$.

2.2. You are given that $X(t)$ is a stochastic process with differential $dX = \alpha X\,dt + \beta X\,dW$. Use Ito's formula to deduce that the stochastic process $Y(t) = 1/X(t)$ has a differential with drift and volatility similarly linear in Y.
Solution: $dY = (-\alpha + \beta^2)Y\,dt - \beta Y\,dW$.

Chapter 3

The Fokker–Planck equation describes the probability distribution

This chapter is otherwise blank.

Chapter 4

Stochastic integration proves Ito's formula

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Chapter 5

Introducing numerical methods for SDEs

Chapter 1 defined the solution of the Ito SDE

$$dX = a(t, X)dt + b(t, X)dW. \tag{5.1}$$

via the finite difference approximation

$$\Delta X_j = a(t_j, X_j)\Delta t_j + b(t_j, X_j)\Delta W_j \tag{5.2}$$

for $\Delta W \sim N(0, \Delta t_j)$. Then, with care, in the limit $\Delta t_j \to 0$, the finite difference approximation $X_j \to X(t_j)$, the Ito SDE solution. A simple numerical scheme is to use (5.2) for some small finite time steps, $\Delta t_j = h$ say.

Unfortunately, the error in (5.2) is poor: the numerical $X_j = X(t_j) + \mathcal{O}(\sqrt{h})$. Other numerical schemes have better errors at the cost of coding more computation.

This chapter develops some theory behind more accurate methods such as this scheme related to the deterministic Improved Euler (Heun) method: for the SDE (5.1) and time steps h compute

$$k_1 = ha(t_j, X_j) + (\Delta W_j - S_j\sqrt{h})b(t_j, X_j),$$

$$k_2 = ha(t_{j+1}, X_j + k_1) + (\Delta W_j + S_j\sqrt{h})b(t_{j+1}, X_j + k_1),$$
$$X_{j+1} = X_j + \tfrac{1}{2}(k_1 + k_2). \tag{5.3}$$

As usual choose $\Delta W_j = \sqrt{h}Z_j$ where random $Z_j \sim N(0,1)$; also choose $S_j = \pm 1$ 'randomly' and independent of the increment ΔW_j. Our challenge in this chapter is to see that this and other numerical schemes do indeed approximate the solution of the Ito SDE (5.1), and to determine the order of error, which for this scheme (5.3) is $\mathcal{O}(h)$ (Theorem 5.11).

Corollary 5.12 establishes that the scheme (5.3) with $S = 0$ (instead of $S = \pm 1$) solves the Stratonovich interpretation of the SDE (5.2) to errors $\mathcal{O}(h)$.

Kloeden [2] elaborates another introduction to numerical methods for SDEs.

5.1 Iterated integrals create numerical approximations

Example 5.1 (Multiple integrals improve accuracy) Consider the SDE $dX = X\,dW$. This SDE is shorthand for the Ito integral

$$X_t = X_0 + \int_0^t X_s\,dW_s\,, \tag{5.4}$$

Over a small time interval $\Delta t = h$ this integral gives $X_h = X_0 + \int_0^h X_t\,dW_t$. Use this as the starting point for an iteration to provide successively more accurate approximations to X_h.

1. Substitute (5.4) into the integrand:

$$\begin{aligned} X_h &= X_0 + \int_0^h X_t\,dW_t \\ &= X_0 + \int_0^h X_0 + \int_0^t X_s\,dW_s\,dW_t \end{aligned}$$

$$
\begin{aligned}
&= X_0 + \int_0^h X_0 \, dW_t + \int_0^h \int_0^t X_s \, dW_s \, dW_t \\
&= X_0 + X_0 \, \Delta W + \text{remainder}.
\end{aligned}
$$

Ignoring the remainder gives the classic Euler step for SDE (5.4):

$$
X_h = X_0 + X_0 \, \Delta W \,. \tag{5.5}
$$

2. Substitute (5.4) into the integrand of the remainder:

$$
\begin{aligned}
X_h &= X_0 + X_0 \, \Delta W + \int_0^h \int_0^t X_0 + \int_0^s X_r \, dW_r \, dW_s \, dW_t \\
&= X_0 + X_0 \, \Delta W + \int_0^h \int_0^t X_0 \, dW_s \, dW_t \\
&\quad + \int_0^h \int_0^t \int_0^s X_r \, dW_r \, dW_s \, dW_t \\
&= X_0 + X_0 \, \Delta W + X_0 \left[\tfrac{1}{2} (\Delta W)^2 - \tfrac{1}{2} h \right] \\
&\quad + \text{remainder}.
\end{aligned}
$$

Ignoring the remainder gives the Milstein approximation for SDE (5.4):

$$
X_h = X_0 + X_0 \, \Delta W + X_0 \left[\tfrac{1}{2} (\Delta W)^2 - \tfrac{1}{2} h \right] \,. \tag{5.6}
$$

Observe it has the drift term.

3. Substitute (5.4) into the integrand of the remainder:

$$
\begin{aligned}
X_h &= X_0 + X_0 \, \Delta W + X_0 \left[\tfrac{1}{2} (\Delta W)^2 - \tfrac{1}{2} h \right] \\
&\quad + \int_0^h \int_0^t \int_0^s X_0 + \int_0^r X_q \, dW_q \, dW_r \, dW_s \, dW_t \\
&= X_0 + X_0 \, \Delta W + X_0 \left[\tfrac{1}{2} (\Delta W)^2 - \tfrac{1}{2} h \right] \\
&\quad + \int_0^h \int_0^t \int_0^s X_0 \, dW_r \, dW_s \, dW_t
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^h\int_0^t\int_0^s\int_0^r X_q \, dW_q \, dW_r \, dW_s \, dW_t \\
&= X_0 + X_0 \, \Delta W + X_0 \left[\tfrac{1}{2}(\Delta W)^2 - \tfrac{1}{2}h\right] \\
&\quad + X_0 \left[\tfrac{1}{6}(\Delta W)^3 - \tfrac{1}{2}h\Delta W\right] + \text{remainder.}
\end{aligned}
$$

Ignoring the remainder gives the next numerical approximation for the SDE (5.4):

$$
\begin{aligned}
X_h =& X_0 + X_0 \, \Delta W + X_0 \left[\tfrac{1}{2}(\Delta W)^2 - \tfrac{1}{2}h\right] \\
&+ X_0 \left[\tfrac{1}{6}(\Delta W)^3 - \tfrac{1}{2}h\Delta W\right] .
\end{aligned} \tag{5.7}
$$

4. One could continue this iteration indefinitely in this simple example giving successively more accurate numerical approximations (they involve Hermite polynomials). The key is the repeated use of the Ito formula to transform the integrand.

Figure 5.1 plots the three approximate numerical schemes for one realisation of the SDE (5.4). The approximations do seem to converge to the exact $X(t)$.

∎

Example 5.2 (General noise integrals) Now consider the more general SDE

$$
dX = b(X)dW . \tag{5.8}
$$

Over a small time interval $\Delta t = h$ this SDE is shorthand for the integral

$$
X_h = X_0 + \int_0^h b(X_t) \, dW_t .
$$

Use this integral as the starting point for an iteration to provide successively more accurate approximations to X_h. But now the integrand is a nonlinear function of the process X_t so we need the stochastic chain rule for functions of a stochastic process:

$$
d\big[f(X(t))\big] = f' \, dX + \tfrac{1}{2} f'' \, dX^2 + \cdots
$$

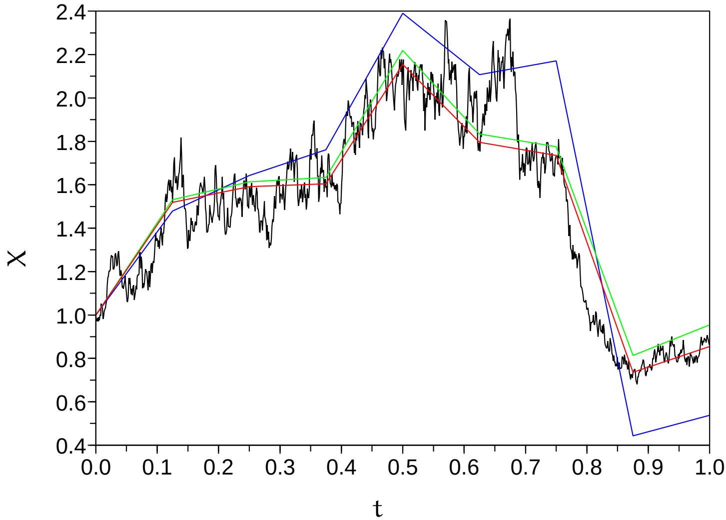

Figure 5.1. *one realisation of the example* SDE (5.4) *with the three different numerical schemes for* $\Delta t = h = 1/8$ *: blue, Euler scheme* (5.5)*; green, Milstien scheme* (5.6)*; and red, next order scheme* (5.7)*.*

$$\begin{aligned} &= f'b\,dW + \tfrac{1}{2}f''b^2\,dW^2 + \cdots \\ &= f'b\,dW + \tfrac{1}{2}f''b^2\,dt \end{aligned}$$

upon remembering that, in effect, '$dW^2 = dt$'. For the 'pure at heart', Ito's integral formula is more precise:

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)b(X_s)\,dW_s + \int_0^t \tfrac{1}{2}f''(X_s)b^2(X_s)\,ds \\ &= f_0 + \int_0^t f_s' b_s\,dW_s + \int_0^t \tfrac{1}{2}f_s'' b_s^2\,ds\,. \end{aligned} \tag{5.9}$$

1. Substitute (5.9) into the integrand of $X_h = X_0 + \int_0^h b(X_t)\,dW_t$

with the choice $f = b$:

$$\begin{aligned} X_h &= X_0 + \int_0^h b(X_t)\,dW_t \\ &= X_0 + \int_0^h \left\{ b_0 + \int_0^t b_s' b_s\,dW_s + \int_0^t \tfrac{1}{2} b_s'' b_s^2\,ds \right\} dW_t \\ &= X_0 + \int_0^h b_0\,dW_t + \int_0^h\int_0^t b_s' b_s\,dW_s\,dW_t + \int_0^h\int_0^t \tfrac{1}{2} b_s'' b_s^2\,ds\,dW_t \\ &= X_0 + b_0\,\Delta W + \text{remainder} \end{aligned}$$

Ignoring the two integrals forming the remainder gives the classic Euler step for the SDE (5.8).

2. Substitute (5.9) into the first integrand of the remainder with the choice $f = b'b$:

$$\begin{aligned} X_h &= X_0 + b_0\,\Delta W \\ &\quad + \int_0^h\int_0^t \left\{ b_0' b_0 + \int_0^s (b_r' b_r)' b_r\,dW_r + \int_0^s \tfrac{1}{2} (b_r' b_r)'' b_r^2\,dr \right\} dW_s\,dW_t \\ &\quad + \int_0^h\int_0^t \tfrac{1}{2} b_s'' b_s^2\,ds\,dW_t \\ &= X_0 + b_0\,\Delta W + \int_0^h\int_0^t b_0' b_0\,dW_s\,dW_t \\ &\quad + \int_0^h\int_0^t\int_0^s (b_r' b_r)' b_r\,dW_r\,dW_s\,dW_t + \int_0^h\int_0^t \tfrac{1}{2} b_s'' b_s^2\,ds\,dW_t \\ &\quad + \int_0^h\int_0^t\int_0^s \tfrac{1}{2} (b_r' b_r)'' b_r^2\,dr\,dW_s\,dW_t \\ &= X_0 + b_0\,\Delta W + b_0' b_0 \left[\tfrac{1}{2} (\Delta W)^2 - \tfrac{1}{2} h \right] + \text{remainder} \end{aligned}$$

Ignoring the three integrals forming the remainder gives the Milstein step for the SDE (5.8):

$$X_1 = X_0 + b_0\,\Delta W + b_0' b_0 \left[\tfrac{1}{2} (\Delta W)^2 - \tfrac{1}{2} h \right]. \tag{5.10}$$

3. Evidently this successive refinement may be continued, but with horribly convoluted integrals.

∎

Order of error The Ito isometry bounds the variance of the remainder. Since the remainder has zero mean and is independent from step to step, then the sum of the errors grows like $\sqrt{n}$ (the variance of the sum is the sum of the variances).

Lemma 5.3. *The Ito integrals* $\int_a^b f\,dW$ *and* $\int_c^d g\,dW$ *are* independent *when the domains of integration do not overlap, that is, when* $a < b \leq c < d$ *or* $c < d \leq a < b$ *.*

Proof. Here consider only piecewise constant stochastic integrand f and g. Then the result follows for more general integrands by similar stochastic limiting arguments to those in Chapter 4. For the two partitions of the two Ito integrals $\int_a^b f\,dW = \sum_j f_j\Delta W_j$ and $\int_c^d g\,dW = \sum_k g_k\Delta W_k$. Thus the expectation

$$\begin{aligned}
&\operatorname{E}\left[\int_a^b f\,dW \times \int_c^d g\,dW\right] \\
&= \operatorname{E}\left[\left(\sum_j f_j\Delta W_j\right) \times \left(\sum_k g_k\Delta W_k\right)\right] \\
&= \operatorname{E}\left[\sum_{j,k} f_j\Delta W_j\, g_k\Delta W_k\right] \\
&= \sum_{j,k} \operatorname{E}\left[f_j\Delta W_j\, g_k\right] \operatorname{E}\left[\Delta W_k\right]
\end{aligned}$$

since for domains $a < b \leq c < d$ the increment ΔW_k is independent of of the other stochastic quantities in each term. But

$E[\Delta W_k] = 0$ and so the entire sum is zero. Consequently, the correlation $E\left[\int_a^b f\,dW \times \int_c^d g\,dW\right] = 0$ which implies the two integrals are stochastically independent. □

Example 5.4 (Reformulate without derivatives) The Milstein scheme (5.10) contains a derivative that we may not know. Rewrite as

$$X_1 = X_0 + \Delta W\, b_0 + \tfrac{1}{2}(\Delta W + \sqrt{h})(\Delta W - \sqrt{h})b_0' b_0$$

Recognise that the derivative term may be approximated by

$$b_0' b_0 \approx \frac{b[X_0 + (\Delta W \pm \sqrt{h})b_0] - b(X_0)}{\Delta W \pm \sqrt{h}}\,.$$

At this stage we are free to choose $\pm$ arbitrarily: we may use this freedom to improve stability, or to reduce error, or something. More research may decide.[1] Let $\tilde{X}_1 = X_0 + (\Delta W \pm \sqrt{h})b_0$ and $\tilde{b}_1 = b(\tilde{X}_1)$, then a derivative free scheme is

$$\begin{aligned} X_1 &= X_0 + \Delta W\, b_0 + \tfrac{1}{2}(\Delta W \mp \sqrt{h})(\tilde{b}_1 - b_0) \\ &= X_0 + \tfrac{1}{2} b_0(\Delta W \pm \sqrt{h}) + \tfrac{1}{2}\tilde{b}_1(\Delta W \mp \sqrt{h}) \\ &= \tfrac{1}{2}\tilde{X}_1 + \tfrac{1}{2}\left[X_0 + (\Delta W \mp \sqrt{h})\tilde{b}_1\right] \end{aligned}$$

Figures 5.2 and 5.3 give one example to show convergence of realisations is $\mathcal{O}(h)$.

Alternatively, we rewrite this scheme as

$$\begin{aligned} k_1 &= (\Delta W \pm \sqrt{h})b(X_0), \\ k_2 &= (\Delta W \mp \sqrt{h})b(X_0 + k_1), \\ X_1 &= X_0 + \tfrac{1}{2}(k_1 + k_2). \end{aligned} \tag{5.11}$$

[1] Empirical numerical experiments suggest to choose $\operatorname{sign}(-\Delta W)$.

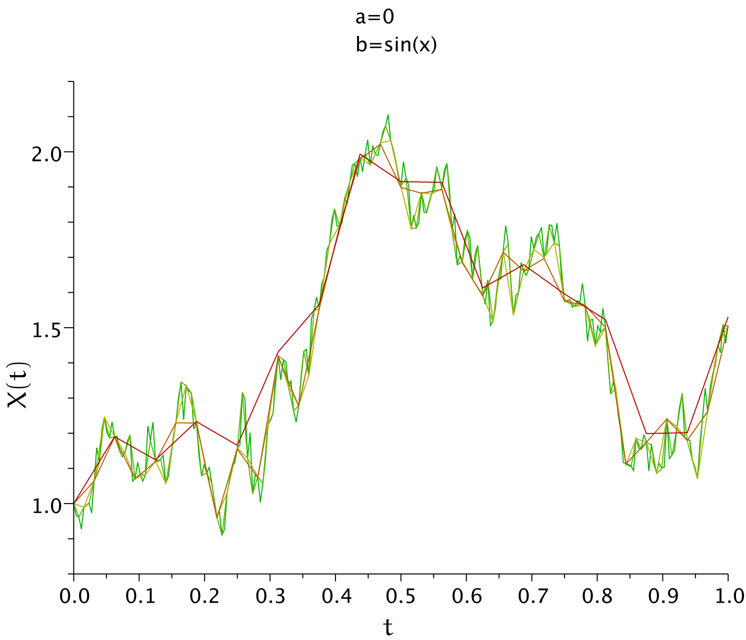

Figure 5.2. *One realisation of the solution to* $dX = \sin(X)dW$, $X(0) = 1$, *with varying sizes of time step.*

This form is an interesting analogy with the deterministic improved Euler method. ∎

Example 5.5 (Stability of derivative free) The derivative free scheme has stability unaffected by the choice of signs. Following Higham [1], apply the scheme to the SDE $dX = \beta X\, dW$, that is, set $b(x) = \beta x$. Consequently $\tilde{X}_1 = X_0[1 + (\Delta W \pm \sqrt{h})\beta]X_0$ and a little algebra then gives the recurrence $X_1 = X_0[1 + \Delta W\beta + \frac{1}{2}(\Delta W^2 - h)\beta^2]$. Since this recurrence is independent of the choice of signs, then the scheme's stability appears independent of the choice. ∎

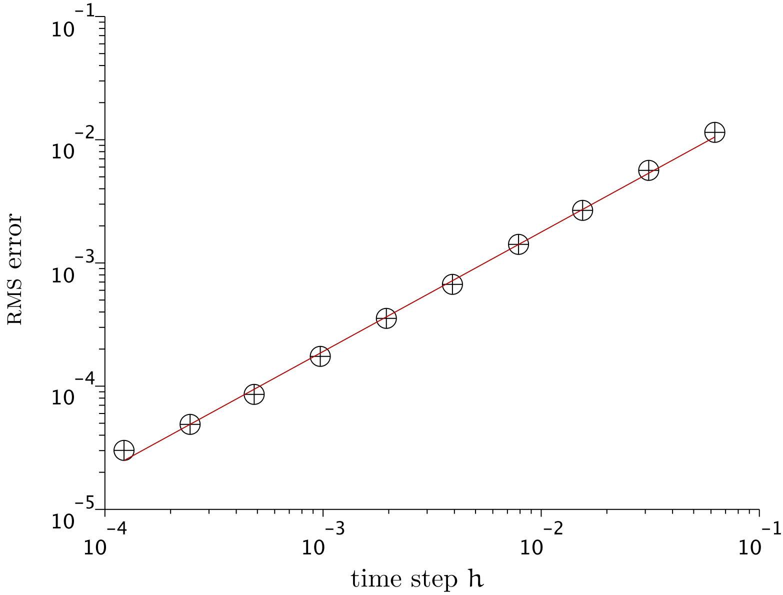

Figure 5.3. *error at time* $t = 1$ *of* 700 *realisations of* $dX = \sin(X)dW$, $X(0) = 1$, *as a function of time step showing the error* $\approx 0.15\,h^{0.97}$ *(the error is estimated by the difference from the result at the smallest time step).*

5.2 A simple Runge–Kutta method avoids derivatives

Now consider a numerical scheme for the general SDE (5.1) that has only one source of noise. With little justification as yet, let's explore the appealing generalisation (5.3) of the scheme (5.11). This scheme appears to be consistent with the Milstein scheme (5.10), and as the

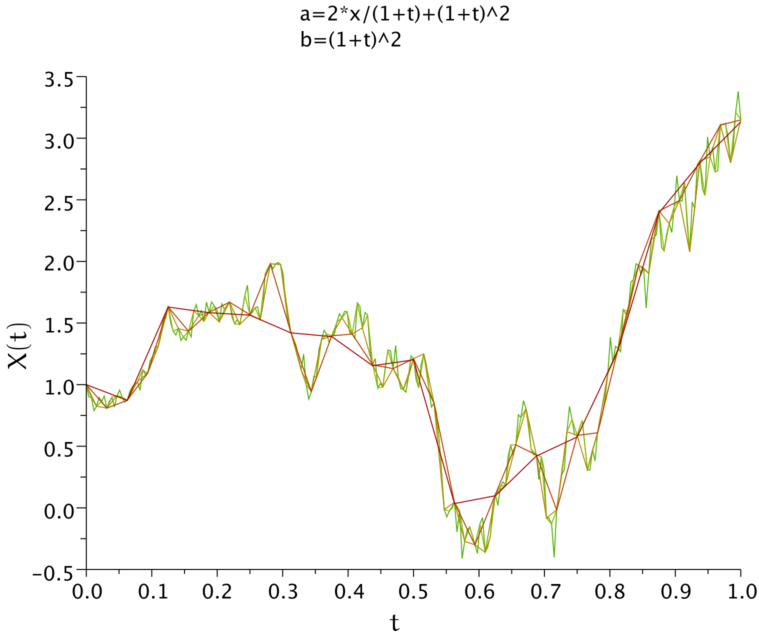

Figure 5.4. *one realisation of the* SDE *in Example 5.6 at different step sizes* h*. The numerical solutions do appear to converge to a solution as step size* $h \to 0$ *.*

volatility $b \to 0$ the scheme reduces to the well known deterministic improved Euler.

Questions?? Does this scheme generalise to vector X?? (I think so) and to multiple noises?? (hmmmm)

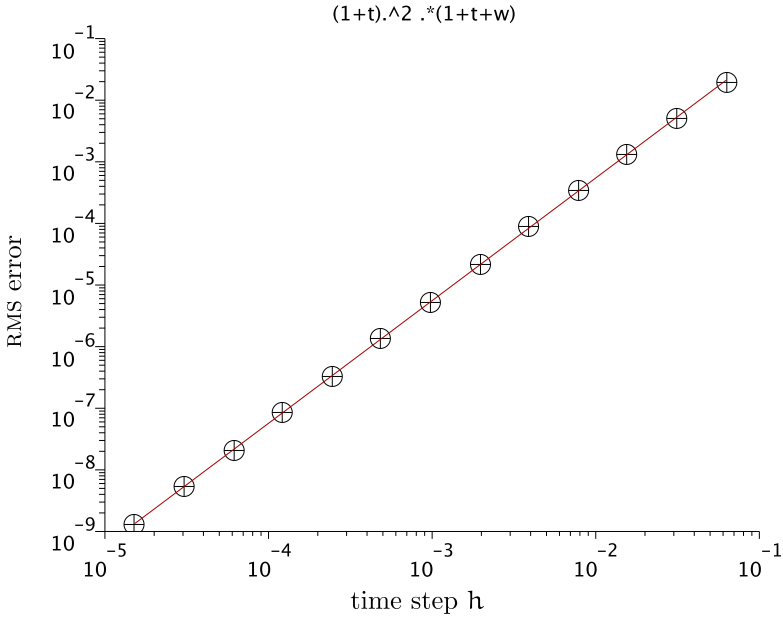

Figure 5.5. *averaging over* 700 *realisations of Example 5.6 at each of many different step sizes shows that, at* $t = 1$, *the error to the analytic solution decreases quadratically, like* h^2.

5.2.1 Empirical evidence indicates $\mathcal{O}(h)$ errors

Example 5.6 Let's try the numerical scheme (5.3) on the SDE

$$dX = \left[\frac{2X}{1+t} + (1+t)^2\right] dt + (1+t)^2 dW .$$

Starting from $X_0 = 1$, Ito's formula shows this SDE has solution $X = (1+t)^2(1+t+W)$. Figure 5.4 shows one realisation of a solution for different time steps. Figure 5.5 plots the RMS relative error in the numerical solutions at time $t = 1$. On the log-log plot it is apparent that the error decreases quadratically: a least squares fit gives[2]

$$\text{RMS error} \approx 5.2\,h^{1.99}.$$

Such rapid quadratic decrease in error is a surprise, generally expect only a linear decrease in error as in the next example. ∎

Example 5.7 Second, let's try the numerical scheme (5.3) on the SDE

$$dX = \left[\tfrac{1}{2}X + \sqrt{1+X^2}\right]dt + \sqrt{1+X^2}\,dW\,.$$

Starting from $X_0 = 0$, Ito's formula shows this SDE has solution $X = \sinh(t+W)$. Figure 5.6 shows one realisation of a solution for different time steps. Figure 5.7 plots the RMS relative error in the numerical solutions at time $t = 1$. The log-log plot indicates the error decreases at the expected linear rate: a least squares fit gives[3]

$$\text{RMS error} \approx 1.34\,h^{1.01}.$$

∎

Example 5.8 Third, let's summarise the accuracy of the numerical scheme (5.3) applied to several more SDEs.

1. RMS error $\approx 0.88\,h^{0.98}$, plotted in Figure 5.8, for SDE $dX = \frac{1}{2}(X-t)dt + (X-t-2)dW$, $X_0 = 3$, with solution $X = 2 + t + \exp(W)$.

[2] Regarding the choice of S in the scheme (5.3): using $S = -\operatorname{sign}(\Delta W)$ gives error $\approx 8.2h^{1.99}$ about 60% bigger; using random S gives between error $\approx 6.9h^{1.99}$; using alternating $S = (-1)^j$ at time step j gives error $\approx 5.6h^{1.99}$; but using one sign all the time is bad, showing a reduction of order, error $\approx 1.9h^{1.45}$ for all plus and error $\approx 4.1h^{1.53}$ for all minus. The smallest error, as reported above, appears to arise from the choice $S = \operatorname{sign}(\Delta W)$.

[3] Regarding the choice of S in the scheme (5.3): using $S = -\operatorname{sign}(\Delta W)$ gives bigger error $\approx 1.76h^{1.00}$; using independently random S gives similar error $\approx 1.45h^{0.99}$; using alternating $S = (-1)^j$ at time step j gives error $\approx 1.66h^{1.01}$; but using one sign all the time is bad, showing a reduction of order, error $\approx 0.41h^{0.50}$ for all plus and error $\approx 0.56h^{0.53}$ for all minus. The smallest error by a little, as reported above, appears to arise from the choice $S = \operatorname{sign}(\Delta W)$.

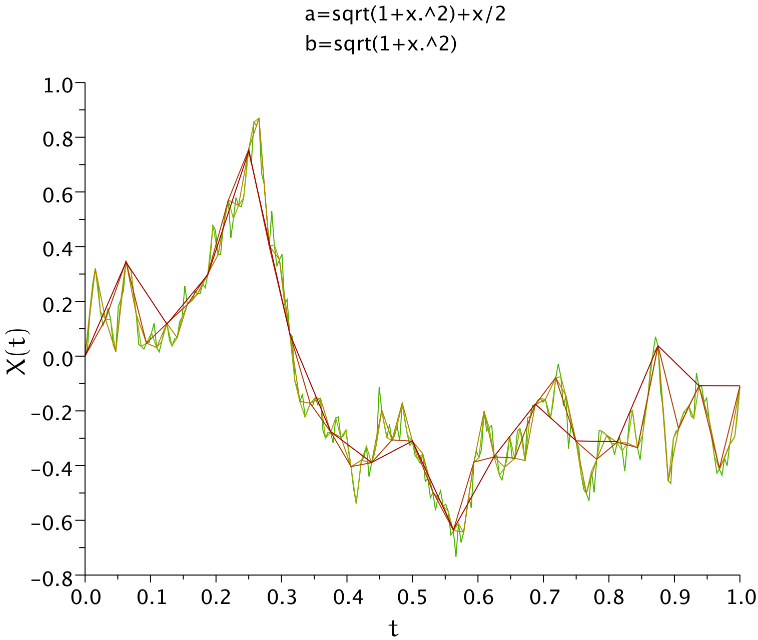

Figure 5.6. *one realisation of the* SDE *in Example 5.7 at different step sizes* h. *The numerical solutions do appear to converge to a solution as step size* $h \to 0$.

2. RMS error $\approx 0.81\, h^{1.01}$, plotted in Figure 5.9, for SDE

$$dX = \left[\frac{X}{1+t} - \tfrac{3}{2}X\left(1 - \frac{X^2}{(1+t)^2}\right)^2\right] dt + (1+t)\left(1 - \frac{X^2}{(1+t)^2}\right)^{3/2} dW\,,$$

$X_0 = 0$, with solution $X = (1+t)W/\sqrt{1+W^2}$.

3. RMS error $\approx 0.06\, h^{2.00}$ for the SDE $dX = -X\, dt + e^{-t} dW$,

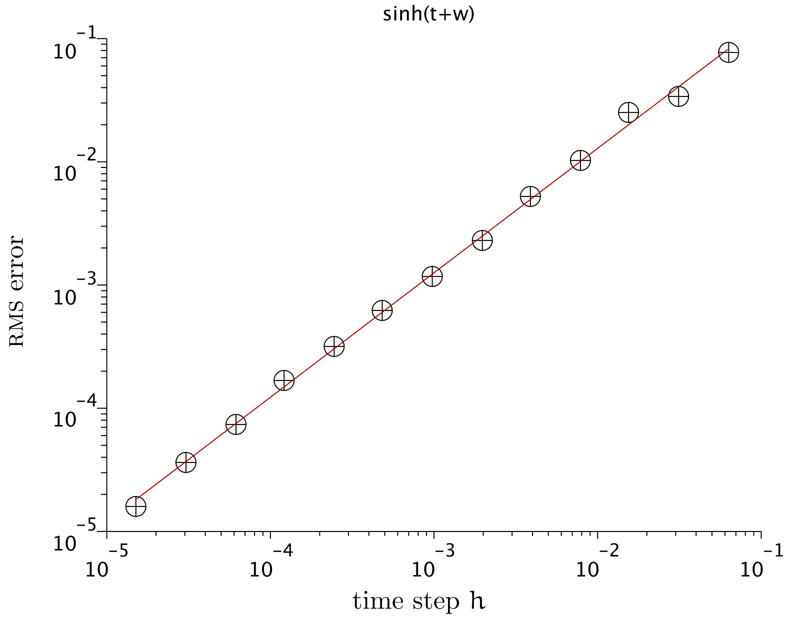

Figure 5.7. *averaging over 700 realisations of Example 5.7 at each of many different step sizes shows that, at $t = 1$, the error to the analytic solution decreases linearly in time step h.*

$X_0 = 0$, with solution $X = e^{-t}W(t)$; additive noise appears to give second order accuracy.

4. RMS error $\approx 0.58\,h^{0.99}$ for the SDE $dX = X\,dW$, $X_0 = 1$, with solution $X = \exp[W(t) - t/2]$.

5. RMS error $\approx 0.36\,h^{1.01}$ for the SDE $dX = -X(1 - X^2)\,dt + (1 - X^2)\,dW$, $X_0 = 0$, with solution $X = \tanh[W(t)]$.

∎

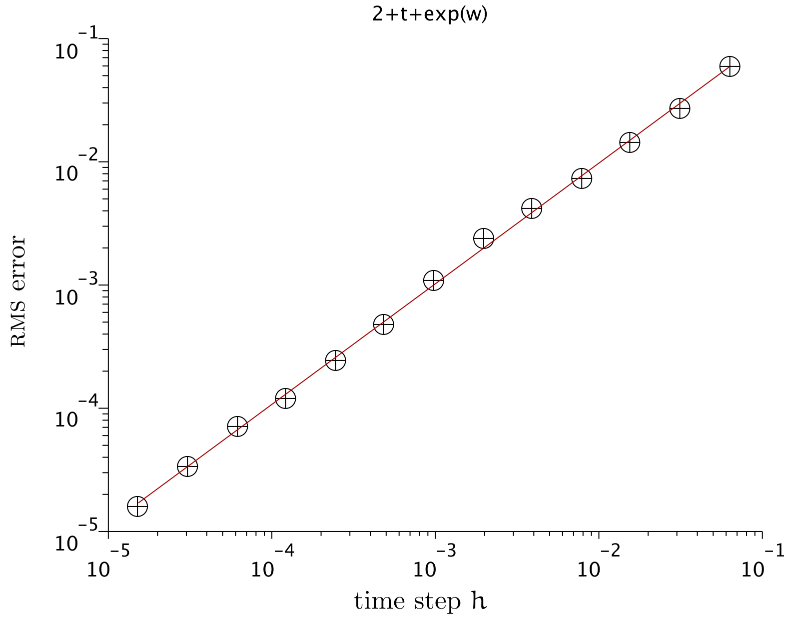

Figure 5.8. *Averaging over* 700 *realisations of Example 5.8.1 at each of many different step sizes shows that, at* $t = 1$, *the error to the analytic solution is of first order in step size* h.

5.2.2 Some theory proves $\mathcal{O}(h)$ accuracy

Proofs that numerical schemes do indeed approximate SDE solutions are often complex. My plan here is to elaborate successively more complicated cases, with the aim that you develop a feel for the analysis before it gets too complex. The first lemma proves that the Runge–Kutta like scheme (5.3) approximates the simplest Ito integrals $X = \int_a^b b(t)\,dW$ to first order in the time step. Second, we explore linear SDEs with additive noise and identify a class of SDEs when the scheme (5.3) is of second order.

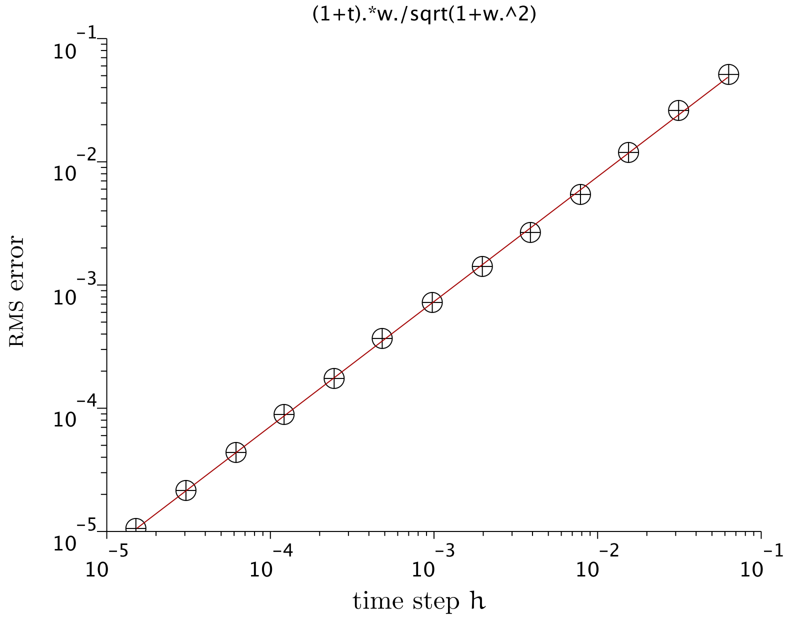

Figure 5.9. *Averaging over* 700 *realisations of Example 5.8.2 at each of many different step sizes shows that, at* $t = 1$, *the* RMS *error to the analytic solution is of first order in step size* h.

One outcome of this section is to precisely 'nail down' the requisite properties of the choice of signs S_j in the scheme (5.3).

Lemma 5.9. *The Runge–Kutta like scheme* (5.3) *has* $\mathcal{O}(h)$ *errors when applied to* $dX = b(t)dW$ *for functions* $b(t)$ *twice differentiable.*

Proof. Without loss of generality, start with the time step from $t_0 = 0$ to $t_1 = t_0 + h = h$. Applied to the very simple SDE $dX =$

$b(t)dW$ the scheme (5.3) computes

$$k_1 = (\Delta W - S\sqrt{h})b_0\,, \quad k_2 = (\Delta W + S\sqrt{h})b_1\,,$$

and then estimates the change in X as

$$\Delta\hat{X} = \tfrac{1}{2}(b_0 + b_1)\Delta W + \tfrac{1}{2}(b_1 - b_0)S\sqrt{h}\,,$$

where $b_0 = b(0)$ and $b_1 = b(h)$. In comparison we use the classic polynomial approximation theorem [3, p.800, e.g.] to relate this to the exact integral. Here write the integrand as the linear interpolant with remainder:

$$b(t) = \tfrac{1}{2}(b_1 + b_0) + \tfrac{1}{h}(b_1 - b_0)(t - h/2) + \tfrac{1}{2}t(t-h)b''(\tau)$$

for some $0 \leq \tau(t) \leq h$. Then the exact change in $X(t)$ is

$$\Delta X = \int_0^h b(t)dW = \tfrac{1}{2}(b_1 + b_0)\Delta W + \tfrac{1}{h}(b_1 - b_0)\int_0^h (t - h/2)dW + \tfrac{1}{2}\int_0^h t(t-h)b''(\tau)dW\,. \tag{5.12}$$

That is, the true integral change $\Delta X = \Delta\hat{X} + \epsilon_0$ where the error

$$\epsilon_0 = \frac{b_1 - b_0}{h}\left[-\tfrac{1}{2}Sh^{3/2} + \int_0^h (t - h/2)dW\right] + \tfrac{1}{2}\int_0^h t(t-h)b''(\tau)dW\,.$$

How big is this error? Take expectations to see that $E[\epsilon_0] = 0$ generally provided $E[S] = 0$. *Thus the choice of signs S in the scheme (5.3) cannot be fixed non-zero: the signs S must be chosen randomly with mean zero.*

Now compute the variance to see the size of the fluctuations in the error. $\operatorname{Var}[\epsilon_0] = E[\epsilon_0^2]$. Look at various contributions in turn. First $E[(Sh^{3/2})^2] = h^3\,E[S^2] = \mathcal{O}(h^3)$. Provided we chose the signs S independently of the noise W then there are no correlations between the S terms and the other two terms. Second,

$$\begin{aligned} \mathrm{E}\left[\left(\int_0^h (t-h/2)\,dW\right)^2\right] &= \int_0^h (t-h/2)^2\,dt \quad \text{by Ito isometry} \\ &= \tfrac{1}{12}h^3 = \mathcal{O}\big(h^3\big). \end{aligned}$$

Third, by the Ito isometry

$$\begin{aligned} \mathrm{E}\left[\left(\int_0^h t(t-h)b''(\tau)\,dW\right)^2\right] &= \int_0^h t^2(t-h)^2 b''(\tau)^2\,dt \\ &\le B_2^2\int_0^h t^2(t-h)^2\,dt = \frac{1}{30}B_2^2h^5, \end{aligned}$$

when the second derivative is bounded, $|b''(t)| \le B_2$. Lastly, the correlation between these previous two integrals is small as

$$\begin{aligned} &\left|\mathrm{E}\left[\int_0^h (t-h/2)\,dW \int_0^h t(t-h)b''(\tau)\,dW\right]\right| \\ &= \left|\int_0^h (t-h/2)t(t-h)b''(\tau)\,dt\right| \\ &\le B_2\int_0^h |(t-h/2)t(t-h)|\,dt = \mathcal{O}\big(h^4\big). \end{aligned}$$

Hence the local error is dominated by the first two contributions and has $\mathrm{Var}[\epsilon_0] = \mathcal{O}\big(h^3\big)$.

Now take $n = \mathcal{O}\big(1/h\big)$ time steps, then the scheme (5.3) approximates the correct solution with error $\epsilon = \sum_{j=0}^{n-1}\epsilon_j$. Firstly, $\mathrm{E}[\epsilon] = 0$ as $\mathrm{E}[\epsilon_j] = 0$ for all time steps. Secondly, as the errors on each time step are independent, the variance

$$\mathrm{Var}[\epsilon] = \sum_{j=0}^{n-1}\mathrm{Var}[\epsilon_j] = n\,\mathrm{Var}[\epsilon_0] = \mathcal{O}\big(nh^3\big) = \mathcal{O}\big(h^2\big).$$

Thus, for the SDE $dX = b(t)dW$, the scheme (5.3) has global error of size $\mathcal{O}(h)$. ∎

This second lemma addresses a little more general SDEs. It not only serves as a 'stepping stone' to a full theorem, but illustrates two other interesting properties. Firstly, we discover a class of SDEs for which the scheme (5.3) is second order accurate in the time step. Secondly, the proof highlights that the sign S in the scheme (5.3) relates to sub-step properties of the noise.

Lemma 5.10. *The Runge–Kutta like scheme* (5.3) *has errors* $\mathcal{O}(h)$ *when applied to the additive noise, linear* SDE $dX = a(t)X\,dt + b(t)dW$ *for functions* a *and* b *twice differentiable. Further, in the exact differential case when* $ab = db/dt$ *(a solution to the* SDE *is then* $X = b(t)W$*) the errors are* $\mathcal{O}(h^2)$.

Proof. In this case, the first step in the scheme (5.3) predicts the change

$$\begin{aligned}\Delta X = h\tfrac{1}{2}(a_0 + a_1)X_0 + \tfrac{1}{2}h^2 a_0 a_1 X_0 + \tfrac{1}{2}(b_0 + b_1)\Delta W \\ + \tfrac{1}{2}a_1 b_0 h(\Delta W - S\sqrt{h}) + \tfrac{1}{2}S\sqrt{h}\Delta b\,. \end{aligned} \tag{5.13}$$

Like before, $a_0 = a(0)$, $a_1 = a(h)$, $b_0 = b(0)$ and $b_1 = b(h)$. We compare this approximate change over the time step h with the true change using iterated integrals. For simplicity use subscripts to denote dependence upon 'times' t, s and r:

$$\Delta X = \int_0^h a_t X_t \, dt + \int_0^h b_t \, dW_t$$

substituting $X_t = X_0 + \Delta X$ inside the integral gives

$$= \int_0^h a_t \left[X_0 + \int_0^t a_s X_s \, ds + \int_0^t b_s \, dW_s \right] dt + \int_0^h b_t \, dW_t$$

$$= X_0 \int_0^h a_t \, dt + \int_0^h a_t \int_0^t a_s X_s \, ds \, dt$$

$$+ \int_0^h a_t \int_0^t b_s \, dW_s \, dt + \int_0^h b_t \, dW_t$$

substituting $X_s = X_0 + \Delta X$ inside the integral gives

$$= X_0 \int_0^h a_t \, dt + \int_0^h a_t \int_0^t a_s \left[X_0 + \int_0^s a_r X_r \, dr + \int_0^s b_r \, dW_r \right] ds \, dt$$
$$+ \int_0^h a_t \int_0^t b_s \, dW_s \, dt + \int_0^h b_t \, dW_t$$
$$= X_0 \int_0^h a_t \, dt + X_0 \int_0^h a_t \int_0^t a_s \, ds \, dt + \int_0^h a_t \int_0^t a_s \int_0^s a_r X_r \, dr \, ds \, dt$$
$$+ \int_0^h a_t \int_0^t a_s \int_0^s b_r \, dW_r \, ds \, dt + \int_0^h a_t \int_0^t b_s \, dW_s \, dt + \int_0^h b_t \, dW_t \,.$$

Replace the negligible triple integrals by their orders of magnitude where, with perhaps some abuse of notation, I use $\mathcal{Q}(h^p)$ to denote random variables of mean zero and variance h^{2p}.[4] The previous proof looked closely at the variances of error terms; here we simplify by focussing only upon their order of magnitude. Then

$$\Delta X = X_0 \int_0^h a_t \, dt + X_0 \int_0^h a_t \int_0^t a_s \, ds \, dt + \mathcal{O}(h^3)$$
$$+ \mathcal{Q}(h^{5/2}) + \int_0^h a_t \int_0^t b_s \, dW_s \, dt + \int_0^h b_t \, dW_t \,. \qquad (5.14)$$

Consider separately the integrals in (5.14). Terms of $\mathcal{O}(h^3)$ or of $\mathcal{Q}(h^{5/2})$ are errors and only tracked by their order of magnitude in h.

- Firstly, $X_0 \int_0^h a_t \, dt = X_0 h \frac{1}{2}(a_0 + a_1) + \mathcal{O}(h^3)$ by the classic trapezoidal rule. This matches the first component in the numerical (5.13).

[4] The reason to separately identify such random errors is that when summed over $\mathcal{O}(1/h)$ *independent* time steps, the central limit theorem implies a local error $\mathcal{Q}(h^p)$ becomes a global error $\mathcal{Q}(h^{p-1/2})$. In comparison, a local error $\mathcal{O}(h^p)$, when summed, becomes a global error $\mathcal{O}(h^{p-1})$.

- Secondly, using the linear interpolation $a_t = a_0 + \frac{\Delta a}{h} t + \mathcal{O}(t^2)$, where as usual $\Delta a = a_1 - a_0$, the repeated integral

$$\begin{aligned}\int_0^h a_t \int_0^t a_s \, ds \, dt &= \int_0^h \left(a_0 + \frac{\Delta a}{h} t\right)\left(a_0 t + \frac{\Delta a}{2h} t^2\right) + \mathcal{O}(t^3) \, dt \\ &= \int_0^h a_0^2 t + a_0 \frac{3\Delta a}{2h} t^2 + \mathcal{O}(t^3) \, dt \\ &= \tfrac{1}{2} a_0^2 h^2 + a_0 \frac{\Delta a}{2} h^2 + \mathcal{O}(h^4) \\ &= \tfrac{1}{2} a_0 a_1 h^2 + \mathcal{O}(h^4)\end{aligned}$$

  Multiplied by X_0, this double integral matches the second term in the numerical (5.13).

- Thirdly, from the proof of the previous lemma, equation (5.12) gives

$$\int_0^h b_t \, dW_t = \tfrac{1}{2}(b_1 + b_0)\Delta W + \frac{\Delta b}{h} \int_0^h \left(t - \tfrac{h}{2}\right) dW_t + \mathcal{Q}(h^{5/2}). \tag{5.15}$$

  The first term here matches the third term in the numerical (5.13). The second term on the right-hand side is an integral remainder that will be dealt with after the next item.

- Lastly, change the order of integration in the double integral

$$\begin{aligned}&\int_0^h a_t \int_0^t b_s \, dW_s \, dt \\ &= \int_0^h b_s \int_s^h a_t \, dt \, dW_s \\ &= \int_0^h b_s \int_s^h a_1 + \mathcal{O}(h - t) \, dt \, dW_s \\ &= \int_0^h b_s a_1 (h - s) + \mathcal{O}\big((h - s)^2\big) \, dW_s\end{aligned}$$

$$\begin{aligned}
&= \int_0^h b_0 a_1 (h-t) + \mathcal{O}\big(h^2\big)\, dW_t \\
&= \int_0^h b_0 a_1 (h-t)\, dW_t + \mathcal{Q}\big(h^{5/2}\big) \\
&= \int_0^h \tfrac{h}{2} b_0 a_1 + b_0 a_1 \big(\tfrac{h}{2} - t\big)\, dW_t + \mathcal{Q}\big(h^{5/2}\big) \\
&= \tfrac{1}{2} h b_0 a_1 \Delta W - b_0 a_1 \int_0^h \big(t - \tfrac{h}{2}\big)\, dW_t + \mathcal{Q}\big(h^{5/2}\big)
\end{aligned}$$

The first term here matches the first part of the fourth term in the numerical (5.13). The second term on the right-hand side is an integral remainder that will be dealt with next.

Hence we now identify that the difference between the Runge–Kutta like step (5.13) and the change (5.14) in the true solution is the error

$$\begin{aligned}
\epsilon_0 = {} & -\tfrac{1}{2} a_1 b_0 h^{3/2} S + \tfrac{1}{2} S \sqrt{h} \Delta b + b_0 a_1 \int_0^h \big(t - \tfrac{h}{2}\big)\, dW_t \\
& - \frac{\Delta b}{h} \int_0^h \big(t - \tfrac{h}{2}\big) dW_t + \mathcal{O}\big(h^3\big) + \mathcal{Q}\big(h^{5/2}\big) \\
= {} & \left[\tfrac{1}{2} S h^{3/2} - \int_0^h \big(t - \tfrac{h}{2}\big) dW_t \right] \left\{ -a_1 b_0 + \frac{\Delta b}{h} \right\} \\
& + \mathcal{O}\big(h^3\big) + \mathcal{Q}\big(h^{5/2}\big) \qquad (5.16)
\end{aligned}$$

Two cases arise corresponding to the main and the provisional parts of lemma 5.10.

- In the general case, the factor in brackets in (5.16) determines the order of error. Choosing the signs S randomly with mean zero, the leading error is then $\mathcal{O}\big(h^3\big) + \mathcal{Q}\big(h^{3/2}\big)$. This is the local one step error. Summing over $\mathcal{O}\big(1/h\big)$ time steps gives that the global error is $\mathcal{O}\big(h^2\big) + \mathcal{Q}\big(h\big)$. That is, the error due to the noise dominates and is generally first order in h as the variance is of order h^2.

However, as the noise decreases to zero, $b \to 0$, the order of error transitions to the deterministic case of $\mathcal{O}(h^2)$.

- The second case is when the factor in braces in (5.16) is small: this occurs for the integrable case $ab = db/dt$ as then the term in braces is $\mathcal{O}(h)$ so that the whole error (5.16) becomes $\mathcal{O}(h^3)+\mathcal{Q}(h^{5/2})$. Again this is the local one step error. Summing over $\mathcal{O}(1/h)$ time steps gives that the global error is $\mathcal{O}(h^2)+\mathcal{Q}(h^2)$. That is, in this case the error is of second order in time step h, both through the deterministic error and the variance of the stochastic errors. Figure 5.5 shows another case when the error is second order.

This concludes the proof. □

Interestingly, we would decrease the size of the factor in brackets in the error (5.16) by choosing the sign S to cancel as much as possible the integral $\int_0^h \left(t - \frac{h}{2}\right) dW_t$. This sub-step integral is one characteristic of the sub-step structure of the noise, and is independent of ΔW. If we knew this integral, then we could choose the sign S to cause some error cancellation; however, generally we do not know the sub-step integral. A converse view is that no matter how we choose signs S, provided the mean is zero, there will be many realisations of the noise, with fixed ΔW, for which some cancellation of the term in brackets occurs. The error is smaller for these realisations.

For example, if one used Brownian bridges to successively refine the numerical approximations for smaller and smaller time steps, then it may be preferable to construct a Brownian bridge compatible with the signs S used on the immediately coarser step size.[5]

[5] The Brownian bridge stochastically interpolates a Wiener process to half-steps in time if all one knows is the increment ΔW over a time step h. The Brownian bridge asserts that the change over half the time step, $h/2$, is $\frac{1}{2}\Delta W - \frac{1}{2}\sqrt{h}Z$ for some $Z \sim N(0,1)$; the change over the second half of the time step

Theorem 5.11. *The Runge–Kutta like scheme* (5.3) *generally has errors* $\mathcal{O}(h)$ *when applied to the* SDE (5.1) *for sufficiently smooth drift and volatility functions* $a(t,x)$ *and* $b(t,x)$*.*

Proof. Straightforward algebra and Taylor series in the smooth coefficients $a(t,x)$ and $b(t,x)$ shows the first step in the scheme (5.3) predicts the change

$$\begin{aligned}\Delta X = a_0h + b_0\Delta W + \tfrac{1}{2}b_0b_0'(\Delta W^2 - S^2h)\\ + \tfrac{1}{2}(\Delta W - S\sqrt{h})\left[hb_0a_0' + \tfrac{1}{2}(\Delta W^2 - S^2h)b_0^2b_0''\right]\\ + h(\Delta W + S\sqrt{h})(\dot{b}_0 + a_0b_0') + \mathcal{O}(h^2)\,,\end{aligned} \tag{5.17}$$

using here an overdot to denote $\partial/\partial t$ and dashes to denote $\partial/\partial x$. Provided the signs S satisfy $S^2 = 1 + \mathcal{O}(h) + \mathcal{Q}(\sqrt{h})$ the first line of (5.17) matches the leading order parts of the solution of the SDE: easiest is to choose $S = \pm 1$ as specified earlier. The remaining identified terms in (5.17) are errors $\mathcal{Q}(h^{3/2})$ provided the signs S have mean $\mathcal{O}(\sqrt{h})$: we normally assume a zero mean for simplicity.

- I suggest choosing signs $S = \pm 1$ randomly, mean zero, independent of ΔW (then the *sample* signs over a global interval would effectively have a local 'sample' mean $\mathcal{Q}(\sqrt{h})$).

  Limited numerical experiments suggests errors reduce with $S = \operatorname{sign}(\Delta W)$. Such dependence upon ΔW appears acceptable provided $E(S\Delta W^2) = 0$.

- You could choose signs $S = \pm 1$ *alternately* from one time step to the next: then the error term in (5.17) effectively becomes finite differences in time of the coefficients that it multiplies, such a difference operator is effectively $\mathcal{O}(\sqrt{h})$ when acting

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is correspondingly $\frac{1}{2}\Delta W + \frac{1}{2}\sqrt{h}Z$. Factoring out the half, these sub-steps are $\frac{1}{2}(\Delta W \mp Z\sqrt{h})$ which match the factors $(\Delta W \mp S\sqrt{h})$ used by the scheme (5.3): the discrete signs $S = \mp$ have mean zero and variance one just like the normally distributed Z of the Brownian bridge.

on the worst ΔW^2 term and so effectively incur an acceptable local error.

- But choosing signs S uniformly is *not* acceptable as then there would be deterministic local errors $\mathcal{O}(h^{3/2})$ that would accumulate to a global error $\mathcal{O}(\sqrt{h})$.

Many introductions to the numerical solutions of SDEs give the following analysis [2, e.g.]. The following analysis gives a stochastic Taylor series solution to the SDE (5.1) to compare with the numerical prediction (5.17). The stochastic Taylor series analysis starts from the Ito formula: for a stochastic process $X(t)$ satisfyng the SDE (5.1), any function of the process

$$f(t, X_t) = f(0, X_0) + \int_0^t L_s^0 f(s, X_s) ds + \int_0^t L_s^1 f(s, X_s) dW_s\,,$$

where $\quad L_s^0 = \left[\frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \tfrac{1}{2}b^2\frac{\partial^2}{\partial x^2}\right]_{t=s}, \quad L_s^1 = \left[b\frac{\partial}{\partial x}\right]_{t=s}.$

For conciseness we again use subscripts t, s and r to denote evaluation at these times, and $f_t = f(t, X_t)$ as appropriate. Now turn to the SDE (5.1) itself: it represents an integral change over the first time step of

$$\Delta X = \int_0^h a(t, X_t) dt + \int_0^h b(t, X_t) dW_t$$

apply the Ito formula to $a(t, X_t)$ and $b(t, X_t)$

$$= \int_0^h a_0 + \int_0^t L_s^0 a_s\, ds + \int_0^t L_s^1 a_s\, dW_s\, dt$$

$$+ \int_0^h b_0 + \int_0^t L_s^0 b_s\, ds + \int_0^t L_s^1 b_s\, dW_s\, dW_t$$

apply the Ito formula to $L_s^1 b_s$

$$= \int_0^h a_0\, dt + \int_0^h \int_0^t L_s^0 a_s\, ds\, dt + \int_0^h \int_0^t L_s^1 a_s\, dW_s\, dt$$

$$+\int_0^h b_0\,dW_t + \int_0^h\int_0^t L_s^0 b_s\,ds\,dW_t$$
$$+\int_0^h\int_0^t L_0^1 b_0 + \int_0^s L_r^0 L_r^1 b_r\,dr + \int_0^s L_r^1 L_r^1 b_r\,dW_r\,dW_s\,dW_t$$

rearrange in order of magnitude

$$= a_0\int_0^h dt + b_0\int_0^h dW_t + L_0^1 b_0\int_0^h\int_0^t dW_s\,dW_t$$
$$+\left[\int_0^h\int_0^t L_s^1 a_s\,dW_s\,dt + \int_0^h\int_0^t L_s^0 b_s\,ds\,dW_t \right.$$
$$\left. + \int_0^h\int_0^t\int_0^s L_r^1 L_r^1 b_r\,dW_r\,dW_s\,dW_t\right]$$
$$+\left[\int_0^h\int_0^t L_s^0 a_s\,ds\,dt + \int_0^h\int_0^t\int_0^s L_r^0 L_r^1 b_r\,dr\,dW_s\,dW_t\right]$$

either evaluate or replace by order in h

$$= a_0 h + b_0\Delta W + b_0' b_0\tfrac{1}{2}(\Delta W^2 - h) + \mathcal{Q}\big(h^{3/2}\big) + \mathcal{O}\big(h^2\big)$$

Provided signs S are chosen as described before, this expression for the change ΔX matches (5.17) for the numerical scheme. The differences are $\mathcal{Q}(h^{3/2}) + \mathcal{O}(h^2)$. This is the local error. Upon summing over $\mathcal{O}(1/h)$ time steps we then deduce the global error is $\mathcal{O}(h)$ for the scheme (5.3). □

Corollary 5.12 (Stratonovich SDEs). *The Runge–Kutta like scheme* (5.3)*, but setting* $S = 0$ *, has errors* $\mathcal{O}(h)$ *when the* SDE (5.2) *is to be interpreted in the Stratonovich sense.*

Proof. Interpreting the SDE (5.2) in the Stratonovich sense implies solutions are the same as the solutions of the Ito SDE

$$dX = (a + \tfrac{1}{2}bb')dt + b\,dW\,.$$

Apply the scheme (5.3) (with $S = 1$ as appropriate to an Ito SDE), or the analysis of the previous proof, to this Ito SDE. Then, for example, the one step change (5.17) becomes

$$\Delta X = (a_0 + \tfrac{1}{2}b_0b_0')h + b_0\Delta W + \tfrac{1}{2}b_0b_0'(\Delta W^2 - h) + \mathcal{Q}\big(h^{3/2}\big) + \mathcal{O}\big(h^2\big).$$

The component of the deterministic drift term that involves b_0b_0' cancel leaving, in terms of the coefficient functions of the Stratonovich SDE,

$$\Delta X = a_0h + b_0\Delta W + \tfrac{1}{2}b_0b_0'\Delta W^2 + \mathcal{Q}\big(h^{3/2}\big) + \mathcal{O}\big(h^2\big). \tag{5.18}$$

Now apply the scheme (5.3) with $S = 0$ to the Stratonovich SDE: Taylor series expansions obtain the one step numerical prediction as (5.17) upon setting $S = 0$. This one step numerical prediction is the same as (5.18) to the same order of errors. Thus the scheme (5.3) with $S = 0$ solves the Stratonovich interpretation of the SDE (5.2).

□

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