The importance of beings fractal

Prof A. J. Roberts
School of Mathematical Sciences
University of Adelaide
mailto:anthony.roberts@adelaide.edu.au

April 7, 2020

Good science is the ability to look at things in a new way and achieve an understanding that you didn't have before ... It is opening windows on the world ... you perceive a little tiny glimpse of the way the Universe hangs together, which is a wonderful feeling. *Hans Kornberg* [2]

Abstract

Fractal geometry, largely inspired by Benoit Mandelbrot [1] during the sixties and seventies, is one of the great advances in mathematics for two thousand years. Given the rich and diverse power of developments in mathematics and its applications, this is a remarkable claim. Often presented as being just a part of modern chaos theory, fractals are momentous in their own right. Euclid's geometry describes the world around us in terms of points, lines and planes—for two thousand years these have formed the somewhat limited repertoire of basic geometric objects with which to describe the universe. *Fractals immeasurably enhance this world-view by providing a description of much around us that is rough and fragmented—of objects that have structure on many sizes.* Examples include: coastlines, rivers, plant distributions, architecture, wind gusts, music, and the cardiovascular system.

Contents

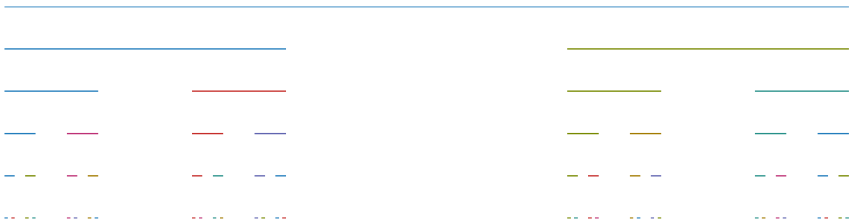

Figure 1: steps in the construction of a Cantor set.

1 Some fractal models

Before discussing in detail the common feature of the previously mentioned examples, in Section 2, I present a few examples of fractals and the type of physical applications that they have.

1.1 Noise and natural events

Have you ever noticed that there are

- some days where nothing goes right?
- times when you just cannot get a decent telephone connection?
- years when drought follows drought?
- long periods when gusts of wind come thick and fast?

That events often occur in bursts is a well documented aspect of the world. The Cantor set[1], as shown in Figure 1, is a model for such bursty phenomena. Construct the Cantor set in the following manner:

1. start with a bar of some length;
2. then remove its middle third to leave two separate thirds;
3. then remove the middle thirds of these to leave four separate ninths;
4. then remove the middle thirds of these to obtain eight separate twenty-sevenths;
5. and so on.

Eventually we just obtain a scattered dust of points. However, this dust is specially distributed into pairs of points, of pairs of pairs of points, and so on. If the original bar represented a time interval, then the dust represents times when events occur and the striking feature is that there are long quiescent periods separating the short bursts of activity that a clump of the points represents.

[1]Cantor was a 19th century mathematician interested in constructing sets with paradoxical properties.

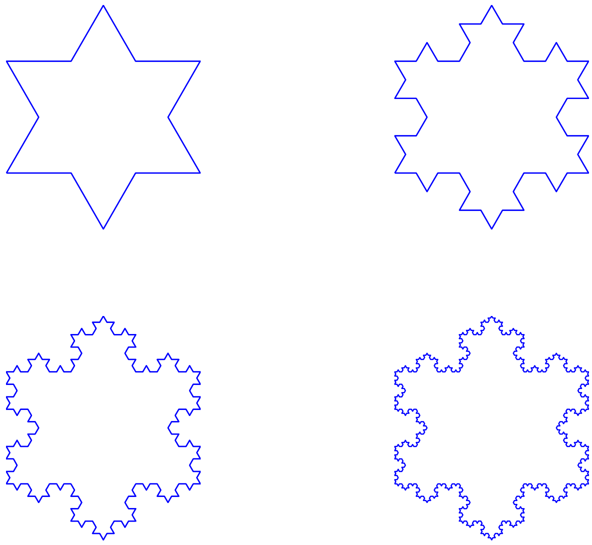

Figure 2: steps in the construction of a Koch island.

1.2 Coastlines and rivers

The line of a coast or the path of a river is tortuous. On a small-scale map of Australia or any other country the coastline has lots of wriggles. Upon examining a larger-scale map the wriggles will be resolved clearly into gulfs and peninsulas. However, many smaller scale wriggles will still be seen. These can only be resolved by looking at an even larger scale map, whereupon they will be seen to be inlets and spits. But once again there will be wriggles in the coast.[2] Similarly for rivers—they exhibit bends and meanders upon many scales of length. The Koch curve shown in Figure 2 models these phenomena.

Starting with an equilateral triangle, we replace the middle third of each side by two segments of the same length (as if we pasted an equilateral triangle of one-third the size onto each side); this forms the second picture above showing large-scale peninsulas and bays. Repeating this process of extracting the middle third of each straight side and replacing it with two segments of the same length, the next stage of the construction gives the third picture; it shows smaller inlets and spits. Continually repeating this process leads to a very wriggly line that is the Koch curve. It is perhaps too convoluted for a coastline, but on the other hand, it looks far more realistic than a routine curve!

[2]L. F. Richardson, see Section 1.3, also was responsible for recognising these fractal characteristics of coastlines.

1.3 Turbulence

Most mathematicians, physicists and engineers would give their right arm to understand what is going on in this picture. It shows something of the highly complex motion that is turbulence in a fluid as expressed by the following ditty[3] by L. F. Richardson [3]:

> Big whorls have little whorls,
> Which feed on their velocity;
> And little whorls have lesser whorls,
> and so on to viscosity.

When air or water moves, a smooth flow quickly breaks up into swirling eddies. These eddies are of a wide range of sizes and, as on a windy day, there are often quiescent periods separating the various wind gusts. The structure of turbulence may be epitomised by a Sierpinski sponge which is formed from building blocks in much the same way as the "Eiffel tower." Form a small unit by putting 20 blocks face to face along the edges and corners of a $3 \times 3 \times 3$ cube, leaving the middle of the six faces and the middle of the cube vacant. Make a bigger unit by connecting 20 of these units together in the same $3 \times 3 \times 3$ pattern. And so on to as large a scale as possible.

2 Scaling and dimensionality

The common theme in these examples is not just that they have detail on many lengths, but also that the structure at any scale is much the same at any other scale—the coastline around a continent looks just like any small part of the coastline. If we take a magnifying glass or microscope to any of these examples then, no matter what the magnification, the geometric detail that we see is the same. This property of looking similar at all scales is termed self-similarity: exact in the artificial examples, and statistical or random in practical applications. In order to tease out the self-similar characteristics of such objects we need to explore the fractal over many lengths and sizes, summarised by equation (1).

The coarsest characteristic of fractals is their dimensionality. While we normally expect a dimension to be an integer, a natural number such as 1, 2 or 3, fractals are best described by means of a dimension which is fractional, such as 1.2 or 0.69. This dimension is obtained by blurring the fractal at some size, counting the number of blobs in this blurred picture, and then seeing how the count varies with the size of the blurring. Another explanation of this process is to count how few "clumps" of a certain size the object can be broken into, and then see how this count varies with the size of the clump.

[3]This ditty is derived from: Big fleas have little fleas upon their backs to bite them, and little fleas have lesser fleas, and so on ad infinitum.

2.1 Points, lines and planes

Lets become familiar with the argument via some well known geometric examples: points, lines, and planes. Consider a line of some length L as shown below. The line could be curved, but for simplicity we take it to be straight. To “blur” the object at a size d I mean that we try to cover as much of the object as possible by discs of diameter d. In this picture I have used $N = 9$ discs to cover the line segment. If the discs were half the diameter, then we would have to use twice as many of them to cover the line; if the discs were one-third the diameter then we would have to use three times as many to cover the line; and so on for other sizes. Typically, the number of discs needed to cover a line of length L is $N = L/d$. The important aspect of this relation is that the number of discs is inversely proportional to the first power of the size (diameter) of the discs: $N \propto d^{-1}$.

Contrast this with what happens when we cover an area A of the plane with discs of some diameter d: $N = 34$ in the above example. Typically, the number of discs needed to cover an area A is inversely proportional to the second power of the size of the discs: $N \propto d^{-2}$; the number would be close to the area, A, divided by the area of each disc, $\pi d^2/4$, to be

$$\frac{4A}{\pi d^2}$$

if it were not for the wastage around the perimeter of each disc.

See that the exponent of this relation between N, the number of discs, and the size of the discs, as measured by the diameter d, is precisely the dimensionality of the object: a line is one-dimensional; an area of the plane is two-dimensional. This relation between the exponent and the dimensionality is true in general. For another example, consider a small number, n, of points distributed in space—for all d smaller than the minimum separation between the points the number of discs needed is precisely the same as the number of points in the set. Thus $N = n \times d^0$, and the 0 exponent matches the zero-dimensionality of a point or a finite number of points.

For the geometric objects introduced earlier, the relation between N and d involves a fractional exponent D:

$$N \propto d^{-D}. \tag{1}$$

It is only reasonable for us to say that the dimensionality of such an object is the fraction D.

References

[1] B. B. Mandelbrot, *The fractal geometry of nature*, 1983.

[2] H. Kornberg, *J. Irreducible Results.*

Table 1: common Euclidean and fractal objects and their fractal dimension.

Object	Dimension
point	0
Cantor set, Section 1.1	0.6309
line	1
Koch curve, Section 1.2	1.2619
plane	2
Sierpinski sponge, Section 1.3	2.7268
solid	3

[3] L. F. Richardson, somewhere and sometime in the 1920s.